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On the concepts of Lie and covariant derivatives of spinors: Part 1

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Abstract. A unified framework for defining Lie and covariant derivatives of spinor fields is presented, which is applicable without restriction on the spacetime connection. The results obtained previously by other authors are analysed and compared with the outcomes of this new formalism.

1. Introduction

The problem of defining the concepts of the Lie and covariant derivatives of a spinor field has already been approached, sometimes in the context of physical applications, by many authors [1–25]. Several formalisms, some of which are inequivalent (e.g. [7] and [23]), exist in the literature. (For a good review, see [22].) However, their common characteristic is that the question is usually posed in special cases, the Lie derivative $\mathcal{L}_X\psi$ of a spinor ψ being defined exclusively when X is a conformal Killing vector, and the covariant derivative $\nabla\psi$, when the connection is compatible with the metric g (in the sense $\nabla g = 0$). It is also sometimes claimed [23, 25] that it is not possible to define, in a geometrically meaningful manner, these notions in cases more general than those just stated.

Recently, however, the problem was analysed again [21], and an attempt was made to define $\mathcal{L}_X\psi$ in full generality by using group-theoretical methods. This shed light on the question, in particular on the existence and uniqueness of the operator $\mathcal{L}_X$, and made contact with some of the existing literature (mainly [5–7, 25]) but it did not provide a framework in which other approaches, such as [23], could be put in perspective.

In the present work, we shall develop precisely such a formalism and we shall show how it generalizes and unifies all the previous results. It should be emphasized that the existence of a general formalism is important not only from the point of view of differential geometry but also for physics, since the covariant derivative of a spinor is essential, for instance, in the formulation of Dirac's equation in curved space [11], whereas the Lie derivative is required to express the spacetime symmetries of theories ([19] and references therein).

The treatment below will be arranged as follows. In section 2, we shall briefly recall Weyl's method [1] for Lie and covariant derivatives since it shows clearly the origin of the problem encountered when one tries to extend to spinor fields the derivatives defined for tensor fields. Then, in sections 3 and 4, we shall introduce the general construction. This will be done in two steps: in section 3, we shall derive the covariant derivative of a

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vector field in a non-standard manner which suggests how to extend the concept of covariant derivative to a spinor field, and this extension itself will be carried out in section 4. Finally, in section 5, we shall analyse the outcomes and compare them with those from the literature.

It should be noted that the Lie and the covariant derivative may be treated in an analogous fashion. Therefore, to avoid unnecessary repetitions, we shall restrict attention to the covariant derivative in sections 2–4. The interpretation of the results in terms of the Lie derivative are collected in the conclusion and in appendix 1.

2. Weyl's method for the covariant derivative

The purpose of this brief section is to illustrate the geometrical reason for the difficulty caused, in general, in defining the parallel transport (and thus the covariant derivative) of a spinor field. The considerations discussed here will be heuristic but they will prove enlightening for the more rigorous developments to be presented later.

Spinors are primarily defined in Minkowski space [24], for which the metric reads

$$g = \eta_{\mu\nu}\,\mathrm{d}x^{\mu} \otimes \mathrm{d}x^{\nu} \qquad \eta_{\mu\nu} \equiv \mathrm{diag}(+1, +1, +1, -1) \tag{2.1}$$

in the natural frame $\mathrm{d}x^{\mu}$ induced by the Cartesian coordinates x^{μ}. In a general four-dimensional manifold $\mathcal{M}$, a (non-holonomic) orthonormal [24] frame $e^{(\hat{\mu})}$ can be chosen, in such a way that, by definition, the metric is expressed as

$$g = \eta_{\mu\nu} e^{(\hat{\mu})} \otimes e^{(\hat{\nu})} \tag{2.2}$$

where the caret over the indices emphasizes that an orthonormal frame is employed. The similarity of (2.2) with (2.1) enables one to extend to manifolds most of the construction of spinors. (As a convenient abuse of terminology, we shall call 'spinor *in* the frame $e^{(\hat{\mu})}$' a spinor obtained by the usual Minkowski-space construction [24] *based on* the orthonormal frame $e^{(\hat{\mu})}$.)

Let ψ be a spinor field. Weyl's method [1] considers that ψ is parallel-transported from a point p to a point q along a vector field X (i.e. along the integral curves of X) iff the components of ψ at q in the orthonormal frame ${}^{\parallel}e_q^{(\hat{\mu})}$ are the same as the components of ψ at p in the orthonormal frame $e_p^{(\hat{\mu})}$, where ${}^{\parallel}e_q^{(\hat{\mu})}$ is the frame obtained by parallel-transporting $e_p^{(\hat{\mu})}$ along X from p to q. The difference between the parallel-transported spinor and the value of the spinor field at q is essentially, in the limit of q tending to p, Weyl's covariant derivative $\nabla_X\psi$. (See [1, 14, 15] for more details.) It is then clear that this geometrical method is inapplicable for an arbitrary non-metric-compatible connection (i.e. a connection satisfying $\nabla g \neq 0$) since a non-metric-compatible connection does not respect the orthonormality of the frame. In other words, ${}^{\parallel}e_q^{(\hat{\mu})}$ is, in general, *not* orthonormal even though $e_p^{(\hat{\mu})}$ is.

One type of non-metricity plays a particular role: the one respecting ortho*gonality* without respecting the *norm* under parallel transport, with the result that an ortho*normal* frame remains ortho*gonal* during parallel transport. This happens when the covariant derivative of the metric along X is proportional to the metric itself:

$$\nabla_X g = 2A(X)g \tag{2.3}$$

where A denotes a one-form. It is then easily seen that Weyl's geometrical method can be adapted to this more general case, which will henceforth be called the 'conformal case'. In the context of the Lie derivative, this generalization is found, for instance, in [23], where the formula analogous to (2.3) is written $\mathcal{L}_X g = kg$. The methods developed in the following section will enable us to go beyond the the restriction (2.3) while reproducing the above-mentioned results of [23] in the special case.

3. Non-standard construction for the covariant derivative of a vector field

This section is devoted to preparing the general formalism for spinorial covariant derivatives presented in section 4. For this study, the most convenient language is that of fibre bundles (see, e.g., [26]). In order to fix the notation, however, we shall begin with some considerations about ordinary tensor calculus. Then, we shall re-interpret the tensorial covariant derivative in terms of bundles. This is well known, and therefore a brief summary will suffice. Finally, we shall develop in more detail a *non-standard* bundle approach to the covariant derivative of a tensor. This method will prove easy to adapt, in section 4, to define the concept of covariant derivative of a spinor.

Let $\mathcal{M}$ be a manifold. Let $e_{(\mu)}$ and $e^{(\mu)}$ denote, respectively, a basis in the tangent space $T\mathcal{M}$ of $\mathcal{M}$ and the dual of this basis. Then, according to tensor calculus, it is well known [27] that the covariant derivative operator ∇ acting on the algebra of tensor fields defined over $\mathcal{M}$ is completely determined by the connection components $\Gamma^{\mu}{}_{\alpha\beta}$ defined as

$$\nabla_{\alpha} e_{(\beta)} \equiv \nabla_{e_{(\alpha)}} e_{(\beta)} \equiv \Gamma^{\gamma}{}_{\beta\alpha} e_{(\gamma)}. \tag{3.1}$$

In general, the connection admits a non-vanishing curvature R, torsion T and non-metricity H. (See appendix 2 for details.) When the basis $e_{(\mu)}$, the metric g, the non-metricity H and the torsion T are given, the connection components are also uniquely determined, and the corresponding expression (B.5) for $\Gamma^{\mu}{}_{\alpha\beta}$ is found in appendix 2. (This is a generalization to $T \neq 0$ and $H \neq 0$ of the theorem [27] stating that there exists a unique metric-compatible, torsion-free connection.)

These connection components $\Gamma^{\mu}{}_{\alpha\beta}$ of tensor calculus may be re-interpreted, in a standard manner [26], using fibre bundles. More precisely, if $PL(\mathcal{M})$ denotes the principal bundle of frames over $\mathcal{M}$, namely the bundle having as fibre over the point p of $\mathcal{M}$ the set of all frames of $T\mathcal{M}$ at p, the structure group (or gauge group) of $PL(\mathcal{M})$ is $GL(4)$, the general linear group in four dimensions. A field of frames above $\mathcal{M}$ is then a section of $PL(\mathcal{M})$, and a vector field on $\mathcal{M}$ is a linear combination of the basic vectors of this section. Moreover, in this context, a connection is defined as a one-form $\mathcal{A}$ over $\mathcal{M}$ with values in the Lie algebra, denoted by $gl(4)$, of the structure group $GL(4)$ and satisfying certain conditions. It is a well known result [26] that it is always possible to obtain such a connection $\mathcal{A}$ in terms of the connection components $\Gamma^{\mu}{}_{\alpha\beta}$ of tensor calculus as

$$\mathcal{A}_{\alpha\beta} = \Gamma_{\alpha\beta\mu} e^{(\mu)} \tag{3.2}$$

where the indices α and β belong to the Lie algebra of the structure group.

The connection $\mathcal{A}$ enables one to define the covariant derivative $\nabla_X e_{(\alpha)}$ of the frame $e_{(\alpha)}$ along a vector field X as [26]

$$\nabla_X e_{(\alpha)} = \mathcal{A}^{\beta}{}_{\alpha}(X) e_{(\beta)} \tag{3.3}$$

and a treatment [26] similar to Weyl's method of section 2 yields the covariant derivative of a vector field V as

$$\nabla_X V = [X(V^{\mu}) + \mathcal{A}^{\mu}{}_{\nu}(X) V^{\nu}] e_{(\mu)}. \tag{3.4}$$

This formula, taken together with (3.2), is identical to the one given by tensor calculus.

With the metric g at our disposal, we can also construct the principal bundle of orthonormal frames (with positive orientation) $PO^{+}(\mathcal{M})$. Its fibre over a point p of $\mathcal{M}$ consists of all orthonormal frames of $T\mathcal{M}$ at p, and its structure group is $SO(3,1)$ if the metric has signature $+++-$. The Lie algebra, denoted by $so(3,1)$, of the gauge group is therefore the algebra of antisymmetric matrices. Consequently the connection components $\Gamma_{\hat{\alpha}\hat{\beta}\hat{\mu}}$ in an orthonormal frame may be re-interpreted as the connection $\mathcal{A}$ of the bundle formalism by

$$\mathcal{A}_{\hat{\alpha}\hat{\beta}} = \Gamma_{\hat{\alpha}\hat{\beta}\hat{\mu}} e^{(\hat{\mu})} \tag{3.5}$$

if and only if $\Gamma_{\hat{\alpha}\hat{\beta}\hat{\mu}}$ is antisymmetric in its first two indices. It is proved in appendix 2 that this is the case if and only if the connection is metric-compatible. If the connection is non-metric, either we return to the bundle $PL(\mathcal{M})$, or we employ the non-standard construction outlined below. The latter approach has the advantage that it is easily adapted to spinors, and so we proceed as follows.

Consider the manifold $\mathcal{M}$ and *two* principal bundles above $\mathcal{M}$: $PL(\mathcal{M})$ and $PO^+(\mathcal{M})$. Given that it is always possible to express an orthonormal frame $e_{(\hat{\mu})}$ of the fibre of $PO^+(\mathcal{M})$ above a point p of $\mathcal{M}$ as a linear combination of a general frame $e_{(\mu)}$ of the fibre of $PL(\mathcal{M})$ above p, there exists a matrix $A^{\beta}{}_{\hat{\alpha}}$ such that

$$e_{(\hat{\alpha})} = e_{(\beta)} A^{\beta}{}_{\hat{\alpha}}. \tag{3.6}$$

This implies that, for every section σ_1 of $PL(\mathcal{M})$ and σ_2 of $PO^+(\mathcal{M})$, there exists a set of matrices A relating σ_1 to σ_2 via (3.6). For convenience, we denote by $A^{\hat{\alpha}}{}_{\beta}$ the inverse matrix of $A^{\beta}{}_{\hat{\alpha}}$. Under the same transformation, vector components change as

$$V^{\hat{\mu}} = A^{\hat{\mu}}{}_{\nu} V^{\nu}. \tag{3.7}$$

Let the connection components of tensor calculus be $\Gamma_{\hat{\alpha}\hat{\beta}\hat{\mu}}$ on an orthonormal frame $e_{(\hat{\mu})}$. On $PO^+(\mathcal{M})$, introduce a connection $\mathcal{A}$ given by

$$\mathcal{A}_{\hat{\mu}\hat{\nu}} \equiv \Gamma_{[\hat{\mu}\hat{\nu}]\hat{\alpha}} e^{(\hat{\alpha})} = (C_{\hat{\mu}\hat{\nu}\hat{\alpha}} + Q_{\hat{\mu}\hat{\nu}\hat{\alpha}} + H_{[\hat{\mu}\hat{\nu}]\hat{\alpha}}) e^{(\hat{\alpha})} \tag{3.8}$$

where the last step follows from (B.7) of appendix 2. This is always possible since, by construction, $\mathcal{A}_{\hat{\mu}\hat{\nu}}$ is antisymmetric. Furthermore, $\mathcal{A}$ leads, by virtue of (3.4), to the following notion of covariant derivative, denoted by ${}^{\mathcal{A}}\nabla$:

$${}^{\mathcal{A}}\nabla_X V \equiv [X(V^{\hat{\mu}}) + \mathcal{A}^{\hat{\mu}}{}_{\hat{\nu}}(X) V^{\hat{\nu}}] e_{(\hat{\mu})} \tag{3.9}$$

for all vector fields X and V.

On $PL(\mathcal{M})$, introduce a connection $\mathcal{B}$ by

$$\mathcal{B}^{\mu}{}_{\nu}(X) \equiv A^{\mu}{}_{\hat{\alpha}} X(A^{\hat{\alpha}}{}_{\nu}) - \tfrac{1}{2} H_{\alpha}{}^{\mu}{}_{\nu} X^{\alpha} \tag{3.10}$$

where H is the non-metricity defined by (B.2) of appendix 2, and A is the matrix appearing in (3.6). It leads to the following notion of covariant derivative, denoted by ${}^{\mathcal{B}}\nabla$:

$${}^{\mathcal{B}}\nabla_X V \equiv [X(V^{\mu}) + \mathcal{B}^{\mu}{}_{\nu}(X) V^{\nu}] e_{(\mu)} \tag{3.11}$$

for all vector fields X and V. If the matrix A of (3.6) is used to express in (3.11) the components V^{μ} in terms of the orthonormal components $V^{\hat{\mu}}$, and $e_{(\mu)}$ in terms of the orthonormal frame $e_{(\hat{\mu})}$, then (3.6), (3.7) and (3.10) imply that (3.11) becomes

$$\begin{aligned} {}^{\mathcal{B}}\nabla_X V &\equiv [X(V^{\hat{\mu}}) + \mathcal{B}^{\hat{\mu}}{}_{\hat{\nu}}(X) V^{\hat{\nu}}] e_{(\hat{\mu})} \\ \mathcal{B}_{\hat{\mu}\hat{\nu}}(X) &\equiv -\tfrac{1}{2} H_{\hat{\alpha}\hat{\mu}\hat{\nu}} X^{\hat{\alpha}} = +\mathcal{B}_{\hat{\nu}\hat{\mu}}. \end{aligned} \tag{3.12}$$

Thus we have at our disposal, by virtue of (3.8), (3.9) and (3.12), two notions of covariant derivative of the orthonormal frame $e_{(\hat{\mu})}$:

$$\begin{aligned} {}^{\mathcal{A}}\nabla_X e_{(\hat{\nu})} &= \eta^{\mu\alpha}(C_{\hat{\alpha}\hat{\nu}\hat{\beta}} + Q_{\hat{\alpha}\hat{\nu}\hat{\beta}} + H_{[\hat{\alpha}\hat{\nu}]\hat{\beta}})X^{\hat{\beta}} e_{(\hat{\mu})} \\ {}^{\mathcal{B}}\nabla_X e_{(\hat{\nu})} &= -\tfrac{1}{2}\eta^{\mu\alpha} H_{\hat{\beta}\hat{\alpha}\hat{\nu}} X^{\hat{\beta}} e_{(\hat{\mu})}. \end{aligned} \tag{3.13}$$

It should be noted that, if the connection is metric-compatible, H vanishes, rendering $\mathcal{B}$ redundant. Our formalism then reduces to the usual one (3.3), (3.5) in this instance.

The important point to emphasize at this stage is that, if we define a new covariant derivative $\nabla_X e_{(\hat{\nu})}$ by

$$\nabla_X e_{(\hat{\nu})} \equiv {}^{\mathcal{A}}\nabla_X e_{(\hat{\nu})} + {}^{\mathcal{B}}\nabla_X e_{(\hat{\nu})} \tag{3.14}$$

the new operator ∇ satisfies

$$\nabla_X e_{(\hat{\nu})} \equiv C^{\hat{\mu}}{}_{\hat{\nu}}(X) e_{(\hat{\mu})} = \Gamma^{\hat{\mu}}{}_{\hat{\nu}\hat{\beta}} X^{\hat{\beta}} e_{(\hat{\mu})} \tag{3.15}$$

which is identical to the answer assigned by tensor calculus to the quantity $\nabla_X e_{(\hat{\mu})}$, even if the connection is non-metric. (When summing the right-hand sides of (3.13) to obtain the connection coefficient appearing in (3.15), use was made of (B.7) of appendix 2.) Finally, as soon as it becomes clear that ∇ operates correctly on the frames, there is no difficulty in obtaining the covariant derivative of a vector field V: one simply adopts (3.4) and replaces $\mathcal{A}$ by C.

In other words, given two sections σ_1 and σ_2 of $PL(\mathcal{M})$ and $PO^+(\mathcal{M})$ respectively, we have introduced on these bundles the two connections $\mathcal{A}$ and $\mathcal{B}$. Each of them possesses its own covariant derivative ${}^{\mathcal{A}}\nabla$ and ${}^{\mathcal{B}}\nabla$. When ${}^{\mathcal{B}}\nabla V$ is transformed into the orthonormal frames of the section σ_2, it induces a value for ${}^{\mathcal{B}}\nabla e_{(\hat{\nu})}$ which has been arranged in such a way that the sum of ${}^{\mathcal{A}}\nabla$ and ${}^{\mathcal{B}}\nabla$ produces the correct expression for the covariant derivative of the orthonormal frame $e_{(\hat{\nu})}$.

Of course, to define the covariant derivative of a vector field V for a general metric-incompatible connection, the above construction, with the antisymmetric part of the connection encoded in $\mathcal{A}$ and the symmetric part, in $\mathcal{B}$, is unnecessarily complicated. It would have been possible to introduce immediately the complete connection in the bundle $PL(\mathcal{M})$ of linear frames without ever requiring the bundle $PO^+(\mathcal{M})$ of orthonormal frames. However, to define spinor fields, as we shall do in the next section, the bundle of orthonormal frames plays a prominent role, and this is why a formalism involving explicitly this bundle is most appropriate in the case of spinors.

4. General covariant derivative of spinor fields

The framework developed in section 3 for covariant derivatives of vector fields will now be adapted to the problem of covariant derivatives of spinor fields. First, we shall briefly define spinor frames and spinor fields. (This will parallel the introduction of the bundle $PO^+(\mathcal{M})$ and of vector fields in section 3.) Then, we shall associate a spinor connection with each of the two connections $\mathcal{A}$ and $\mathcal{B}$ of section 3. This will provide us with a general notion of covariant derivative of a spinor field which will be compared, in section 5, with the alternative definitions available in the literature. As in section 3, fibre bundles are the most convenient language for this study. We emphasize that bundles have been used before (e.g. in [2–9]) to shed light on the question; the novelty of the present approach lies in the use of the decomposition of the connection into the parts $\mathcal{A}$ and $\mathcal{B}$.

Let $PO^+(\mathcal{M})$ be the bundle of orthonormal frames. Its transition functions [28], which belong to its structure group $SO(3,1)$, will be denoted by t_{ij}. On the other hand, consider the spin group $SP(4)$, also denoted [25] by $_\pm\Gamma^+$. For every element s of $SP(4)$, the transformation χ_s,

$$\chi_s(x) \equiv s \vee x \vee s^{-1} \tag{4.1}$$

where $\vee$ means the Clifford product, is an element [25] of $SO(3,1)$, the mapping being two-to-one. The bundle of spin frames $PS^+(\mathcal{M})$ over $\mathcal{M}$ is now defined [28] as the unique principal bundle with transition functions $\tilde{t}_{ij}$ belonging to $SP(4)$ and given by

$$\chi_{\tilde{t}_{ij}} = t_{ij}. \tag{4.2}$$

Because of the double-valuedness of χ_s, the bundle $PS^+(\mathcal{M})$ is a double covering of $PO^+(\mathcal{M})$. Moreover, an assignment of a family of spin frames over $\mathcal{M}$ is defined as a section of $PS^+(\mathcal{M})$, and a spinor field over $\mathcal{M}$, as a linear combination of the elements of the frames of this section. Spin frames will be denoted by $\tilde{e}_{(M)}$, and their duals, by $\tilde{e}^{(M)}$.

The mapping χ between the groups $SP(4)$ and $SO(3,1)$ induces a Lie-algebra homomorphism, denoted by χ_*, between the Lie algebras $sp(4)$ and $so(3,1)$ of these groups. It is possible to prove [2,25] that χ_* is invertible and that the element of the Lie algebra $sp(4)$ associated by χ_*^{-1} with an antisymmetric matrix $F_{\mu\nu}$ of $so(3,1)$ reads, when an orthonormal frame is chosen in the cotangent space $T^*\mathcal{M}$:

$$-8\chi_*^{-1}(F) = F_{\mu\nu}[e^{(\hat{\mu})} \vee e^{(\hat{\nu})} - e^{(\hat{\nu})} \vee e^{(\hat{\mu})}]. \tag{4.3}$$

Consider now an irreducible representation γ of the Lie algebra $sp(4)$ in the space of spinors. (All such representations are equivalent [25].) This means that γ associates with every s belonging to $sp(4)$, a linear operator γ_s transforming a spinor into a spinor, which implies that γ_s may be written

$$\gamma_s \equiv (\gamma_s)^M{}_N \tilde{e}^{(N)} \otimes \tilde{e}_{(M)}. \tag{4.4}$$

In particular, by combining γ with χ_*^{-1} of (4.3), one can associate with every antisymmetric matrix $F_{\mu\nu}$ of $so(3,1)$ an action Δ_F on spinors as

$$\begin{aligned}\Delta_F \equiv \gamma_{\chi_*^{-1}(F)} &= -\tfrac{1}{8} F_{\mu\nu}(\gamma_{e^{(\hat{\mu})}} \circ \gamma_{e^{(\hat{\nu})}} - \gamma_{e^{(\hat{\nu})}} \circ \gamma_{e^{(\hat{\mu})}}) \\ &= -\tfrac{1}{2} F_{\mu\nu}(\sigma^{\hat{\mu}\hat{\nu}})^M{}_N \tilde{e}^{(N)} \otimes \tilde{e}_{(M)}\end{aligned} \tag{4.5}$$

where $\circ$ denotes the composition of mappings, and the following abbreviations have been used:

$$\begin{aligned}&4(\sigma^{\hat{\mu}\hat{\nu}})^M{}_N \equiv (\gamma^{\hat{\mu}})^M{}_P(\gamma^{\hat{\nu}})^P{}_N - (\gamma^{\hat{\nu}})^M{}_P(\gamma^{\hat{\mu}})^P{}_N = -4(\sigma^{\hat{\nu}\hat{\mu}})^M{}_N \\ &(\gamma^{\hat{\mu}})^M{}_N \equiv (\gamma_{e^{(\hat{\mu})}})^M{}_N.\end{aligned} \tag{4.6}$$

These considerations now enable us to define the covariant derivative of a spinor field $\psi \equiv \psi^M \tilde{e}_{(M)}$ as we did in section 3 for a vector field in the non-standard way. As in (3.4), we put

$$\nabla_X \psi = X(\psi^M)\tilde{e}_{(M)} + \mathcal{D}(\psi) \tag{4.7}$$

in which $\mathcal{D}(\psi)$ denotes the action of the connection on ψ. Following the non-standard method, we must construct $\mathcal{D}$ from the two connections $\mathcal{A}$ and $\mathcal{B}$. Moreover, because of the additivity of $\mathcal{A}$ and $\mathcal{B}$ seen in (3.14), we impose

$$\mathcal{D} = \mathcal{D}_{\mathcal{A}} + \mathcal{D}_{\mathcal{B}} \tag{4.8}$$

where $\mathcal{D}_{\mathcal{A}}$ and $\mathcal{D}_{\mathcal{B}}$ describe, respectively, the actions induced by $\mathcal{A}$ and $\mathcal{B}$ on spinors.

Given that $\mathcal{A}_{\hat{\mu}\hat{\nu}}(X)$ of (3.8) is antisymmetric, the operation $\Delta_{\mathcal{A}(X)}$ of (4.5), (4.6) is well defined, and we may put

$$\mathcal{D}_{\mathcal{A}} \equiv \Delta_{\mathcal{A}(X)} = -\tfrac{1}{2}\mathcal{A}_{\hat{\mu}\hat{\nu}}(X)(\sigma^{\hat{\mu}\hat{\nu}})^M{}_N\tilde{e}^{(N)} \otimes \tilde{e}_{(M)}. \tag{4.9}$$

Consequently, by (4.7)–(4.9), the covariant derivative reads

$$\nabla_X\psi = [X(\psi^M) - \tfrac{1}{2}\mathcal{A}_{\hat{\mu}\hat{\nu}}(X)(\sigma^{\hat{\mu}\hat{\nu}})^M{}_N\psi^N]\tilde{e}_{(M)} + \mathcal{D}_{\mathcal{B}}(\psi). \tag{4.10}$$

It is important to emphasize† that $\mathcal{A}_{\hat{\mu}\hat{\nu}}$ of (4.10) does, in general, contain terms coming from the non-metricity H, as it is clear from the contribution $H_{[\hat{\mu}\hat{\nu}]\hat{\alpha}}$ to (3.8), the definition of $\mathcal{A}_{\hat{\mu}\hat{\nu}}$. We are, therefore, *not* constructing the action $\mathcal{D}_{\mathcal{A}}$ of (4.9) by 'lifting' to spinors the *metric-compatible* connection induced by $\mathcal{A}$. Had we done so, the metric-incompatibility H would *not* have appeared in the definition (4.10), as it does (implicitly) through the definition (3.8) of $\mathcal{A}_{\hat{\mu}\hat{\nu}}$.

If we now turn to the construction of $\mathcal{D}_{\mathcal{B}}$ of (4.8), we see that a definition similar to (4.9) is not applicable to $\mathcal{B}$ since $\mathcal{B}_{\hat{\mu}\hat{\nu}}$ is symmetric, by virtue of (3.12), and the set of symmetric matrices does not form a Lie algebra. There can thus be no Lie algebra homomorphism between this set and the Clifford algebra $sp(4)$. Some authors (e.g. [5–7]) then put

$$\mathcal{D}_{\mathcal{B}}(\psi) \equiv {}^{\mathrm{K}}\mathcal{D}_{\mathcal{B}}(\psi) = 0 \tag{4.11}$$

in which the superscript K has been used to distinguish this operator $\mathcal{D}_{\mathcal{B}}$ from another one defined below. Because of the symmetry of $\mathcal{B}_{\hat{\mu}\hat{\nu}}$ and the antisymmetry of $\sigma^{\hat{\mu}\hat{\nu}}$, it is equivalent to write

$${}^{\mathrm{K}}\mathcal{D}_{\mathcal{B}} = -\tfrac{1}{2}\mathcal{B}_{\hat{\mu}\hat{\nu}}(X)(\sigma^{\hat{\mu}\hat{\nu}})^M{}_N\tilde{e}^{(N)} \otimes \tilde{e}_{(M)}. \tag{4.12}$$

By combining (4.10) and (4.12), and using (3.15), we obtain the final expression for the covariant derivative ${}^{\mathrm{K}}\nabla_X$:

$${}^{\mathrm{K}}\nabla_X\psi = [X(\psi^M) - \tfrac{1}{2}\Gamma_{\hat{\mu}\hat{\nu}\hat{\alpha}}X^{\hat{\alpha}}(\sigma^{\hat{\mu}\hat{\nu}})^M{}_N\psi^N]\tilde{e}_{(M)}. \tag{4.13}$$

For a general connection, ${}^{\mathrm{K}}\nabla$ is a well-defined operator which reduces to the Weyl covariant derivative if the connection is metric-compatible, and this formula is the one adopted, e.g., in [7] and [21] but is different from that of [23]. (The latter state their results for the problem of the Lie derivative, but the translation between Lie and covariant derivative is elementary and can be found in appendix 1.)

It is, however, possible, in contrast with (4.11), to go further than to 'lift' trivially, to spinors, the action of $\mathcal{B}$ as 0. If the tensor $\mathcal{B}_{\hat{\mu}\hat{\nu}}(X)$ is decomposed into its irreducible parts under the subgroups of $GL(4)$, the finest decomposition [29], which pertains to $SO(3,1)$, is

$$\begin{aligned} &\mathcal{B}_{\hat{\mu}\hat{\nu}}(X) = {}^0\mathcal{B}_{\hat{\mu}\hat{\nu}}(X) + \mathcal{T}(X)\eta_{\mu\nu} \\ &{}^0\mathcal{B}_{\hat{\mu}\hat{\nu}}(X) \equiv \mathcal{B}_{\hat{\mu}\hat{\nu}}(X) - \mathcal{T}(X)\eta_{\mu\nu} \qquad 4\mathcal{T}(X) \equiv \eta^{\alpha\beta}\mathcal{B}_{\hat{\alpha}\hat{\beta}}(X). \end{aligned} \tag{4.14}$$

† The authors would like to thank the referee for drawing their attention to the appropriateness of insisting on this point.

The explicit expressions for the trace-free part ${}^0\mathcal{B}_{\hat{\mu}\hat{\nu}}(X)$ and the trace part $\mathcal{T}(X)$ read, after employing (3.12):

$$ {}^0\mathcal{B}_{\hat{\mu}\hat{\nu}}(X) = -\tfrac{1}{2}X^{\hat{\alpha}}H_{\hat{\alpha}\hat{\mu}\hat{\nu}} - \mathcal{T}(X)\eta_{\mu\nu} \qquad -8\mathcal{T}(X) = X^{\hat{\alpha}}H_{\hat{\alpha}}{}^{\hat{\beta}}{}_{\hat{\beta}}. \tag{4.15} $$

The effect of $\mathcal{T}(X)\eta_{\mu\nu}$ on an orthonormal frame $e_{(\hat{\mu})}$ is to respect the orthogonality while creating an expansion of the base-vectors under parallel transport along X. More precisely, by (3.11) the trace part $\mathcal{T}(X)\eta_{\mu\nu}$ acts on an orthonormal frame as

$$ {}^{\mathcal{T}}\nabla_X e_{(\hat{\mu})} = \mathcal{T}(X)e_{(\hat{\mu})}. \tag{4.16} $$

This behaviour is equivalent to (2.3) and corresponds to what was called the 'conformal case' in section 2. On the other hand, ${}^0\mathcal{B}_{\hat{\mu}\hat{\nu}}(X)$ alters the scalar products while respecting the volume, which corresponds to generating a shear of the basis. The 'lift' denoted by ${}^{\mathrm{K}}\mathcal{D}_{\mathcal{B}}$ in (4.11) ignores thus the influence of both expansion and shear on the parallel transport of spinors.

As far as the shear is concerned, there is no natural way of 'lifting' it to spinors other than by 0 since the argument that the symmetric matrices do not form a Lie algebra holds for the shear as well as for the whole of $\mathcal{B}_{\hat{\mu}\hat{\nu}}(X)$ itself. There is, however, a canonical lift of the conformal part $\mathcal{T}$, as emphasized in section 2. If one considers the action (4.16) of $\mathcal{T}$ on an orthonormal frame, it is natural to put, for the spin frame $\tilde{e}_{(M)}$:

$$ {}^{\mathcal{T}}\nabla_X \tilde{e}_{(M)} \equiv \tfrac{1}{2}\mathcal{T}(X)\tilde{e}_{(M)} \tag{4.17} $$

in which the factor $1/2$ has been inserted because of the double-valuedness of the covering of $PO^+(\mathcal{M})$ by $PS^+(\mathcal{M})$.

These considerations lead us to adopt, for $\mathcal{D}_{\mathcal{B}}$ of (4.10), the value

$$ \mathcal{D}_{\mathcal{B}} = \mathcal{D}_{{}^0\mathcal{B}} + \mathcal{D}_{\mathcal{T}} = 0 + \tfrac{1}{2}\mathcal{T}(X)\tilde{e}^{(M)} \otimes \tilde{e}_{(M)} \tag{4.18} $$

where $\tilde{e}^{(M)}\otimes\tilde{e}_{(M)}$ is easily seen as being the identity operator. By virtue of (4.10) and (4.18), we have therefore, as alternative expression to (4.13), the following covariant derivative of a spinor ψ:

$$ \nabla_X\psi = \{X(\psi^M) + [-\tfrac{1}{2}\mathcal{A}_{\hat{\mu}\hat{\nu}}(X)(\sigma^{\hat{\mu}\hat{\nu}})^M{}_N + \tfrac{1}{2}\mathcal{T}(X)\delta^M{}_N]\psi^N\}\tilde{e}_{(M)}. \tag{4.19} $$

After substitution into this expression of the explicit form (4.15) for $\mathcal{T}(X)$ and (3.15) for $\mathcal{A}_{\hat{\mu}\hat{\nu}}(X)$, there follows

$$ \nabla_X\psi = \{X(\psi^M) - [\tfrac{1}{2}\Gamma_{\hat{\mu}\hat{\nu}\hat{\alpha}}X^{\hat{\alpha}}(\sigma^{\hat{\mu}\hat{\nu}})^M{}_N + \tfrac{1}{16}X^{\hat{\alpha}}H_{\hat{\alpha}}{}^{\hat{\beta}}{}_{\hat{\beta}}\delta^M{}_N]\psi^N\}\tilde{e}_{(M)} \tag{4.20} $$

which is our final expression for the covariant derivative of a spinor ψ. It is now in order to comment on it and to compare it in more detail with alternative ones available in the literature. This will now be done in the conclusion.

5. Conclusion

In this work, we presented a new formalism to define the Lie and covariant derivatives of a spinor field in complete generality. We first traced, in section 2, the origin of the difficulty in defining these derivatives to the fact that an orthonormal frame, under Lie or parallel transport, does not remain orthonormal in general. Then, in section 3, we developed a non-standard bundle formulation of covariant differentiation of a vector field which consisted essentially in expressing on the bundle of orthonormal frames the antisymmetric part of the connection and the symmetric part, on the bundle of general frames.

It was possible to 'lift' canonically the antisymmetric part to the bundle of spin frames by exploiting the homomorphism between the algebras of the structure groups of the spin bundle and of the bundle of orthonormal frames. (This had been done before [2–9].) For the symmetric part, it was seen that no such homomorphism could exist, and therefore it was necessary to use a different type of 'lift'.

After decomposing the symmetric part into contributions which are irreducible under $SO(3,1)$ and which may be interpreted as the shear and the expansion due to the operation of parallel transport, it was shown that only the expansion could be 'lifted' canonically to spinors. The final expression for the covariant derivative thus obtained reads

$$\begin{aligned}\nabla_X\psi &= \{X(\psi^M) - [\tfrac{1}{2}\Gamma_{\hat\mu\hat\nu\hat\alpha}X^{\hat\alpha}(\sigma^{\hat\mu\hat\nu})^M{}_N + \tfrac{1}{16}X^{\hat\alpha}H_{\hat\alpha}{}^{\hat\beta}{}_{\hat\beta}\delta^M{}_N]\psi^N\}\tilde e_{(M)}\\ \nabla_X e^{(\hat\mu)} &\equiv -\Gamma^{\hat\mu}{}_{\hat\nu\hat\alpha}X^{\hat\alpha}e^{(\hat\nu)} \qquad \nabla_X g = X^{\hat\alpha}H_{\hat\alpha\hat\mu\hat\nu}e^{(\hat\mu)}\otimes e^{(\hat\nu)}\end{aligned} \tag{5.1}$$

where the definitions of the quantities $\Gamma^{\hat\mu}{}_{\hat\nu\hat\alpha}$ and $H_{\hat\alpha\hat\mu\hat\nu}$ can be proved from (B.2) and (B.3) in appendix 2.

If the connection is metric-compatible, the contribution involving H in (5.1) vanishes, and our definition agrees with Weyl's definition [1] adopted by all the other authors. If the connection is metric-incompatible, the $\sigma^{\hat\mu\hat\nu}$ term in (5.1) is the same as in [7] and [21], but there is, in addition, the 'conformal term' involving $\delta^M{}_N$, not found in [7] or [21].

For the reason explained in section 2, some authors [23] have given special attention to the case $\nabla_X g = 2A(X)g$, for a certain one-form A, which is often claimed to be the most general situation where the covariant derivative has a geometrical meaning [25]. In our formalism, this corresponds to the special value for H

$$H_{\hat\alpha\hat\mu\hat\nu} = 2A_{\hat\alpha}\eta_{\mu\nu}. \tag{5.2}$$

If this expression is substituted into (5.1), our derivative agrees with that of [23] and, of course, differs from that of [7] and [21] by the above-mentioned conformal term.

Even in the case of a general H, i.e. when H is *not* of the form (5.2), an additional remark may be made about our definition (5.1), in relation with (5.2): it is possible to construct, from an arbitrary connection, a unique conformal one. This is done by assuming formally that (5.2) holds and constructing $A_{\hat\alpha}$ formally by inverting (5.2) as

$$8A_{\hat\alpha} = H_{\hat\alpha}{}^{\hat\beta}{}_{\hat\beta}. \tag{5.3}$$

This formal $A_{\hat\alpha}$ may then be substituted in the standard definition of the covariant derivative valid for a conformal connection, for instance in [23]. This is *not* what we are doing here, as it is easily established from (5.1), (5.3) and (B.7). It is also quite obvious that our method is bound to be different from 'mimicking' a conformal connection since the latter approach, based on (5.3), uses *only* the information from the non-metricity $H_{\hat\mu\hat\nu\hat\alpha}$ which is contained in the contraction $H_{\hat\alpha}{}^{\hat\beta}{}_{\hat\beta}$. However, our definition (5.1) uses, *in addition to* this contraction, the contribution $H_{[\hat\mu\hat\nu]\hat\alpha}$ which is (implicitly) present in the antisymmetric part of the connection term $\Gamma_{\hat\mu\hat\nu\hat\alpha}$, as can be seen from (B.7). Of course, when the connection happens to be conformal, we are back to (5.2), and the comments made there apply.

All these considerations can be trivially repeated for the Lie derivative. The development is briefly sketched in appendix 1. It is sufficient to state here the result:

$$\begin{aligned}\mathcal{L}_X\psi &= \{X(\psi^M) + [\tfrac{1}{2}L_{\hat\mu\hat\nu}(\sigma^{\hat\mu\hat\nu})^M{}_N - \tfrac{1}{8}L^{\hat\beta}{}_{\hat\beta}\delta^M{}_N]\psi^N\}\tilde e_{(M)}\\ \mathcal{L}_X e^{(\hat\mu)} &\equiv L^{\hat\mu}{}_{\hat\nu}e^{(\hat\nu)} \qquad \mathcal{L}_X g = (L_{\hat\mu\hat\nu} + L_{\hat\nu\hat\mu})e^{(\hat\mu)}\otimes e^{(\hat\nu)}.\end{aligned} \tag{5.4}$$

The explicit expression for $L_{\hat\mu\hat\nu}$ in terms of X is found in appendix 1.

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Appendix 1. Results for the Lie derivative

As announced in the main text, the problem of the Lie derivative is almost identical to that of the covariant derivative, provided a similar formalism is used in both cases. Following the conventions of [19], we define the Lie derivative by

$$\mathcal{L}_X e^{(\hat{\mu})} = L^{\hat{\mu}}{}_{\hat{\nu}} e^{(\hat{\nu})}. \tag{A.1}$$

Such an L always exists since the Lie derivative is type-preserving. It is easy to prove [19] that, in terms of X, this L is given by

$$L^{\hat{\mu}}{}_{\hat{\nu}} = e_{(\hat{\nu})}(X^{\hat{\mu}}) + D^{\hat{\mu}}{}_{\hat{\nu}\hat{\alpha}} X^{\hat{\alpha}} \tag{A.2}$$

where the commutation coefficients D are defined in appendix 2. When (A.1) is compared with $\nabla_X e^{(\hat{\mu})}$ of (5.1), the following 'translation rule' is found to transform covariant derivatives into Lie derivatives:

$$-\Gamma^{\hat{\mu}}{}_{\hat{\nu}\hat{\alpha}} X^{\hat{\alpha}} \longrightarrow L^{\hat{\mu}}{}_{\hat{\nu}}. \tag{A.3}$$

Furthermore, (A.1) implies that $\mathcal{L}_X g$ has the value

$$\mathcal{L}_X g = (L_{\hat{\mu}\hat{\nu}} + L_{\hat{\nu}\hat{\mu}}) e^{(\hat{\mu})} \otimes e^{(\hat{\nu})} \tag{A.4}$$

which, together with the expression for $\nabla_X g$ of (5.1), yields the second 'translation rule'

$$X^{\hat{\alpha}} H_{\hat{\alpha}\hat{\mu}\hat{\nu}} \longrightarrow L_{\hat{\mu}\hat{\nu}} + L_{\hat{\nu}\hat{\mu}}. \tag{A.5}$$

The formula (A.4) also implies that X is a Killing vector if and only if L is antisymmetric. (Details are avilable in [19].) When the two rules (A.3) and (A.5) are applied to the definition (5.1) of $\nabla_X \psi$, one finds, for the Lie derivative, the definition given by (5.4).

It should be noted that the literature seldom uses the quantity $L_{\mu\nu}$ of (A.1) but expresses the Lie derivative in terms of the covariant derivative [5–7, 21, 25]. This confuses the issue since it might seem to imply that the Lie derivative depends on the connection Γ through the covariant derivative. More precisely, some authors [5–7, 21, 25] give the following definitions (in the absence of torsion and non-metricity):

$$\begin{aligned} {}^*\mathcal{L}_X \psi &\equiv {}^*\nabla_X \psi + \tfrac{1}{2} X_{\hat{\mu};\hat{\nu}} (\sigma^{\hat{\mu}\hat{\nu}})^M{}_N \psi^N \tilde{e}_{(M)} \\ {}^*\nabla_X \psi &\equiv X(\psi^M) \tilde{e}_{(M)} - \tfrac{1}{2} \Gamma_{\hat{\mu}\hat{\nu}\hat{\alpha}} X^{\hat{\alpha}} (\sigma^{\hat{\mu}\hat{\nu}})^M{}_N \psi^N \tilde{e}_{(M)} \end{aligned} \tag{A.6}$$

where the semi-colon denotes the tensorial covariant derivative, and the asterisk on the left has been employed to avoid ambiguities with our operators of Lie and covariant derivative. If equations (A.6) are adopted in the general case, one proves easily, after substituting the explicit form (A.2) for L and (B.5) of appendix 2 for $\Gamma_{\hat{\mu}\hat{\nu}\hat{\alpha}}$, that

$$[X(\psi^M) + \tfrac{1}{2} L_{\hat{\mu}\hat{\nu}} (\sigma^{\hat{\mu}\hat{\nu}})^M{}_N \psi^N] \tilde{e}_{(M)} = {}^*\mathcal{L}_X \psi + \tfrac{1}{2} (H_{[\hat{\mu}\hat{\nu}]\hat{\alpha}} - T_{[\hat{\mu}\hat{\nu}]\hat{\alpha}}) X^{\hat{\alpha}} (\sigma^{\hat{\mu}\hat{\nu}})^M{}_N \psi^N e_{(M)} \tag{A.7}$$

from which the relationship between ${}^*\mathcal{L}_X \psi$ and our definition $\mathcal{L}_X \psi$ can be determined.

Appendix 2. General connections

An arbitrary connection Γ put on a manifold exhibits, in general, torsion (T), curvature (R) and non-metricity (H). If ∇, $e_{(\mu)}$ and $e^{(\mu)}$ denote, respectively, the covariant derivative, an arbitrary basis in the tangent space and the dual of this basis, one may define the relevant quantities as

$$T(X,Y) \equiv \nabla_X Y - \nabla_Y X - [X,Y] \equiv T^{\gamma}{}_{\alpha\beta} e^{(\alpha)} \otimes e^{(\beta)} \otimes e_{(\gamma)} \tag{B.1}$$

$$H(Z,X,Y) \equiv (\nabla_Z g)(X,Y) \equiv H_{\alpha\beta\gamma} e^{(\alpha)} \otimes e^{(\beta)} \otimes e^{(\gamma)} \tag{B.2}$$

$$\begin{aligned} &\nabla_\alpha e_{(\beta)} \equiv \nabla_{e_{(\alpha)}} e_{(\beta)} \equiv \Gamma^{\gamma}{}_{\beta\alpha} e_{(\gamma)} \\ &[e_{(\alpha)}, e_{(\beta)}] \equiv D^{\gamma}{}_{\alpha\beta} e_{(\gamma)} \end{aligned} \tag{B.3}$$

where X, Y, Z are vector fields and g is the metric. (The definition of the curvature R will not be necessary for what follows and is therefore not reproduced here.)

These relations imply:

$$\begin{aligned} &\Gamma_{\alpha\gamma\beta} - \Gamma_{\alpha\beta\gamma} = T_{\alpha\beta\gamma} + D_{\alpha\beta\gamma} \\ &\Gamma_{\alpha\beta\gamma} + \Gamma_{\beta\alpha\gamma} = e_{(\gamma)}(g_{\alpha\beta}) - H_{\gamma\alpha\beta} \\ &\Gamma_{\alpha\beta\gamma} \equiv g_{\alpha\mu}\Gamma^{\mu}{}_{\beta\gamma}. \end{aligned} \tag{B.4}$$

Those, in turn, yield the explicit expression of the connection Γ in terms of g, T, H and D:

$$\begin{aligned} &\Gamma_{\alpha\beta\gamma} \equiv \langle\alpha\beta\gamma\rangle + Q_{\alpha\beta\gamma} - K_{\alpha\beta\gamma} \\ &\langle\alpha\beta\gamma\rangle \equiv [\alpha\beta\gamma] + C_{\alpha\beta\gamma} \qquad \text{(Levi–Civita connection)} \\ &[\alpha\beta\gamma] \equiv \Sigma(e_{(\gamma)}(g_{\alpha\beta})) = +[\alpha\gamma\beta] \qquad \text{(Christoffel symbol)} \\ &C_{\alpha\beta\gamma} \equiv \Sigma(D_{\gamma\alpha\beta}) = -C_{\beta\alpha\gamma} \qquad \text{(non-holonomicity)} \\ &Q_{\alpha\beta\gamma} \equiv \Sigma(T_{\gamma\alpha\beta}) = -Q_{\beta\alpha\gamma} \qquad \text{(contorsion tensor)} \\ &K_{\alpha\beta\gamma} \equiv \Sigma(H_{\gamma\alpha\beta}) = +K_{\alpha\gamma\beta} \qquad \text{(non-metric part)} \end{aligned} \tag{B.5}$$

in which the symbol Σ applied to a three-index object $W_{\gamma\alpha\beta}$ is defined as

$$2\Sigma(W_{\gamma\alpha\beta}) \equiv W_{\gamma\alpha\beta} + W_{\beta\alpha\gamma} - W_{\alpha\beta\gamma}. \tag{B.6}$$

For the purposes of the construction of the general covariant derivative of a spinor field, it is convenient to decompose the connection coefficients $\Gamma_{\alpha\beta\mu}$ of (B.5) in terms of their symmetric and antisymmetric parts $\Gamma_{(\alpha\beta)\mu}$ and $\Gamma_{[\alpha\beta]\mu}$ respectively as

$$\begin{aligned} &2\Gamma_{(\alpha\beta)\mu} \equiv \Gamma_{\alpha\beta\mu} + \Gamma_{\beta\alpha\mu} = e_{(\mu)}(g_{\alpha\beta}) - H_{\mu\alpha\beta} \\ &\Gamma_{[\alpha\beta]\mu} \equiv \tfrac{1}{2}(\Gamma_{\alpha\beta\mu} - \Gamma_{\beta\alpha\mu}) = -e_{([\alpha)}(g_{\beta]\mu}) + C_{\alpha\beta\mu} + Q_{\alpha\beta\mu} + H_{[\alpha\beta]\mu}. \end{aligned} \tag{B.7}$$

In the special case where an orthonormal frame is selected at each point, the metric is constant and therefore drops out of (B.7) completely. In addition, the symmetric part $\Gamma_{(\alpha\beta)\mu}$ vanishes then iff the connection is metric-compatible. It is important to emphasize that, to distinguish tensorial components in an orthonormal frame from those in an arbitrary one, the former will be surmounted by a caret. Thus, $\Gamma_{\hat\alpha\hat\beta\hat\mu}$ denotes the orthonormal components of the connection, whereas $\Gamma_{\alpha\beta\mu}$ denotes arbitrary components.

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